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Equations of state for an infinite-ranged Ising spin glass in the framework of the Parisi solution

A V Goltsev

A F Ioffe Physical–Technical Institute, Academy of Sciences of the USSR, Leningrad, 194021, USSR

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Abstract. For the Parisi solution of an infinite-ranged Ising spin glass we derive a self-consistent system of equations which determine the free energy, magnetisation and the function $q(x)$ for $H \geqslant 0$. Using these equations we prove the existence of massless modes and carry out a comparison with results of Sompolinsky's dynamic approach.

1. Introduction

In the framework of the Edwards–Anderson mean field theory (Edwards and Anderson 1975) it is supposed that the replica symmetry is unbroken. Unfortunately, this mean field theory gives a negative entropy at low temperatures for the SK model (Sherrington and Kirkpatrick 1975, 1978) which can be solved in the mean field approximation. Investigating the stability of the SK solution below the transition temperature (T_c), de Almeida and Thouless (1978, hereafter referred to as AT) have found that the replica symmetry is broken in the spin glass phase at low magnetic field. A new mean field theory for spin glasses has been proposed by Parisi (1979, 1980a, b, c). In the framework of the Parisi solution the local order parameter is a function ($q(x)$) defined on the interval [0, 1]. If the replica symmetry is unbroken, $q(x)$ becomes constant and we recover the Edwards–Anderson mean field theory. If the replica symmetry is broken, as happens in the spin glass phase at low magnetic fields, $q(x)$ is x dependent. The function $q(x)$ can be obtained by maximising an effective free energy $F[q(x)]$:

$$F = \max_{q(x)} F[q(x)]. \tag{1}$$

The expansion of the functional $F[q(x)]$ in powers of $q(x)$ may be obtained for T near T_c. Retaining only the term of order $q^4(x)$ in $F[q(x)]$, Parisi (1980a, c) has found the function $q(x)$ for T close to T_c. At low temperatures, however, the exact functional $F[q(x)]$ is unknown and so the results obtained by Parisi (1980a, b, c) for some magnetic and thermodynamic properties of spin glasses are only approximate.

In this paper we derive self-consistent equations that enable one to obtain the function $q(x)$, F and magnetisation (m) by a self-consistent procedure at all temperatures $T < T_c$ at non-zero magnetic field (H). We give a solution of these equations for T close to T_c at zero magnetic field (§ 2). In § 3 we prove the existence of massless modes. In § 4 the Parisi solution is compared with results of the dynamic approach (Sompolinsky 1981).

2. Equations of state

In the SK model the free energy per spin (F) is given by

$$\beta F(Q_{\alpha\beta}) = -\tfrac{1}{4}\beta^2 - \lim_{n\to 0}\frac{1}{n}\max\Big\{-\tfrac{1}{4}\beta^2\sum_{\alpha,\beta} Q^2_{\alpha\beta}$$
$$+\ln \operatorname{Tr}\exp\Big(\tfrac{1}{2}\beta^2\sum_{\alpha,\beta} Q_{\alpha\beta}S_\alpha S_\beta + \beta H\sum_\alpha S_\alpha\Big) - 1\Big\} \qquad (2)$$

where the indices α, β run from 1 to n and the trace is over the 2^n values of the $S_\alpha = \pm 1$ and $Q_{\alpha\beta}$ is an $n\times n$ matrix, identically zero on the diagonal ($Q_{\alpha\alpha}=0$). Parisi (1979) proposed the following parametrisation of the matrix $Q_{\alpha\beta}$:

$$Q_{\alpha\beta} = q_i \qquad \text{if } I(\alpha/m_i) \neq I(\beta/m_i) \qquad \text{and} \qquad I(\alpha/m_{i+1}) = I(\beta/m_{i+1}) \qquad (3)$$

where $q_i(i=0,K)$ are real numbers and $m_i(i=1,K)$ are integers such that m_{i+1}/m_i is an integer with $m_0 = 1$ and $m_{K+1}=n$; $I(x)$ is an integer valued function: its value is the smallest integer greater than or equal to x. For $n=0$ the function $q(x)$ is defined by

$$q(x) = q_i \qquad \text{for } m_i > x > m_{i+1} \qquad (i=0,K). \qquad (4)$$

Introducing the quantity

$$G = \operatorname{Tr}\exp\Big(\tfrac{1}{2}\beta^2\sum_{\alpha,\beta} Q_{\alpha\beta}S_\alpha S_\beta + \sum_\alpha h_\alpha S_\alpha\Big) \qquad (5)$$

we find in the limit $n\to 0$

$$m = \langle S_\alpha\rangle = \lim_{n\to 0}\frac{\partial G}{\partial h_\alpha}\Big|_{h_\alpha = H\beta}, \qquad Q_{\alpha\beta} = \langle S_\alpha S_\beta\rangle = \lim_{n\to 0}\frac{\partial^2 G}{\partial h_\alpha\,\partial h_\beta}\Big|_{h_\alpha=\beta H}. \qquad (6)$$

Using a simple method suggested by Duplantier (1981), the evaluation of equations (5) and (6) can be reduced to a solution of some differential equations. For the free energy F, the magnetisation m, and the function $q(x)$ we obtain

$$\beta F = -\tfrac{1}{4}\beta^2\Big(1 + \int_0^1 q^2(x)\,dx - 2q(1)\Big) - [f(0, \beta H + \beta z\sqrt{q(0)})]_z, \qquad (7)$$

$$m = [\varphi(0, \beta H + \beta z\sqrt{q(0)})]_z, \qquad (8)$$

$$q(x) = [\psi_x(0, \beta H + \beta z\sqrt{q(0)})]_z, \qquad (9)$$

where

$$[f(z)]_z \equiv \int_{-\infty}^{\infty}\frac{dz}{\sqrt{2\pi}}\exp(-z^2/2)f(z).$$

The functions $f(y,h)$ and $\varphi(y,h)$, $y\in[0,1]$, satisfy equations

$$\dot f = -\tfrac{1}{2}\beta^2\dot q[f'' + y(f')^2], \qquad (10)$$

$$\dot\varphi = -\tfrac{1}{2}\beta^2\dot q[\varphi'' + 2yf'\varphi'], \qquad (11)$$

$f' \equiv \partial f/\partial h$, $\dot f \equiv \partial f/\partial y$, with the boundary conditions

$$f(1,h) = \ln(2\cosh h), \qquad \varphi(1,h) = \tanh h.$$

It is easy to show that $\varphi(y,h)=f'(y,h)$. The function $\psi_x(x,h)$, $y\in[0,x]$, satisfies equation (11) with the boundary condition

$$\psi_x(x,h)=\varphi^2(x,h).$$

A brief derivation of equations (8)–(11) is presented in the appendix. The equations (7) and (10) were obtained by Parisi (1980b)†. We omit details of our solution of equations (8)–(11) and now present results for the case $H=0$ and T close to $T_c=1$ ($t=1-T/T_c\ll 1$):

$$q(x)=\begin{cases} q(1), & x_1<x\leqslant 1,\\ a_1x+O(x^4), & 0<x\leqslant x_1,\end{cases}$$
$$q(1)=t+t^2-t^3+O(t^4), \tag{12}$$
$$a_1=\tfrac{1}{2}(1+3t)+O(t^3), \qquad x_1=2t-4t^2+10t^3+O(t^4).$$

It is interesting to compare our results (12) with the Sompolinsky (1981) results obtained from the dynamic approach. In the framework of the dynamic approach the Parisi solution $q(x)$ corresponds to $\dot{\Delta}(x)=-x\dot{q}(x)$. The use of equation (12) yields the same relation

$$(1+3t)\Delta(x)+q^2(x)=t^2+2t^3$$

as that obtained by Sompolinsky (1981).

3. Massless modes

In our previous paper (Goltsev 1983) we studied the stability of the Parisi solution and obtained the result that the smallest eigenvalue ($\lambda_{\min}$) of the third family eigenvalues (De Dominicis and Kondor 1983) of the matrix of coefficients ($\partial^2F/\partial Q_{\alpha\beta}\partial Q_{\gamma\nu}$) governing fluctuations about the Parisi solution is equal to

$$\lambda_{\min}=2q(1)-1+T^2-\langle S_\alpha S_\beta S_\gamma S_\nu\rangle, \tag{13}$$

where the replica indices α, β, γ and ν ($\alpha\neq\beta\neq\gamma\neq\nu$) satisfy the condition: $Q_{\alpha\beta}=Q_{\alpha\gamma}=Q_{\alpha\nu}=Q_{\beta\gamma}=Q_{\beta\nu}=Q_{\gamma\nu}=q(1)$. Hereafter the correlation function $\langle S_\alpha S_\beta S_\gamma S_\nu\rangle$ in (13) will be written as $\langle S_\alpha S_\beta S_\gamma S_\nu\rangle_1$. An inequality

$$T^2\geqslant 1-2q(1)+\langle S_\alpha S_\beta S_\gamma S_\nu\rangle_1 \tag{14}$$

should be satisfied for the stability of the Parisi solution. For a full stability analysis it is necessary to prove that eigenvalues of first and second families (De Dominicis and Kondor 1983) are greater than or equal to zero. In the limit $n\to 0$ the right-hand side of (14) may be written in the form (Goltsev 1983)

$$1-2q(1)+\langle S_\alpha S_\beta S_\gamma S_\nu\rangle_1=[\xi(0,\beta H+\beta z\sqrt{q(0)})]_z$$

where the function $\xi(y,h)$ satisfies equation (11) with the boundary condition

$$\xi(1,h)=\operatorname{sech}^4 h. \tag{15}$$

To prove inequality (14) we consider the function $\chi(x)$ defined by

$$\chi(x)=[\chi_x(0,\beta H+\beta z\sqrt{q(0)})]_z \tag{16}$$

† Equations (8)–(11) have also been derived independently by de Almeida and Lage (1983).

where the function $\chi_x(y, h)$, $y \in [0, x]$, obeys equation (11) with the boundary condition

$$\chi_x(x, h) = \varphi'(x, h). \tag{17}$$

Using the recursion relation (A2), we find

$$\chi(x) = \beta(1 - q(1) + \Delta(x)) \tag{18}$$

where $\dot{\Delta}(x) = -x\dot{q}(x)$, $\Delta(1) = 0$. As follows from Sompolinsky (1981) the function $\chi(x)$ is the local susceptibility measured at the frequency $\omega_x = t_x^{-1}$. A calculation of $\dot{\chi}(x)$ for $x = 1$ from equations (16) and (18) yields

$$1 = \beta^2[\xi(0, \beta H + \beta z\sqrt{q(0)})]_z$$

or, equivalently,

$$T^2 = 1 - 2q(1) + \langle S_\alpha S_\beta S_\gamma S_\nu \rangle_1. \tag{19}$$

Therefore $\lambda_{\min} = 0$ and there are massless modes.

Using equations (9) and (11) and the boundary condition for the function $\psi_x(y, h)$, we calculate $\partial\psi_y(x, h)/\partial y$ for $y = x = 0$ and obtain

$$1 = [(\chi_0(0, \beta H + \beta z\sqrt{q(0)}))^2]_z.$$

This relation yields that for $H = 0$ the static (zero field) susceptibility $\chi(0) = 1$.

4. Comparison with the Somopolinsky results

Now we derive the Sompolinsky (1981) results from equations (8)–(11). Using the relation $\varphi(x, h) = f'(x, h)$ the differential equation (11) for $\varphi(x, h)$ may be reduced to an integral equation:

$$\varphi(x, h) = \int_{-\infty}^{\infty} \prod_{x<y<1} \frac{dz(y)}{(2\pi\dot{q}(y))^{1/2}} \exp\left(-\tfrac{1}{2}\int_x^1 dy \frac{z^2(y)}{\dot{q}(y)}\right) \tanh[H_1(x, \{z\})] \tag{20}$$

where

$$H_y(x, \{z\}) = \beta \int_x^y dy_1\, z(y_1) - \beta^2 \int_x^y dy_1\, \dot{\Delta}(y_1)\varphi(y_1, H_{y_1}(x, \{z\})) + h. \tag{21}$$

Substituting $h = H_x(0, \{z\})$ we obtain

$$\varphi(x, H_x(0, \{z\})) = \int_{-\infty}^{\infty} \prod_{x<y<1} \frac{dz(y)}{(2\pi\dot{q}(y))^{1/2}} \exp\left(-\tfrac{1}{2}\int_x^1 dy \frac{z^2(y)}{\dot{q}(y)}\right) \tanh[H_1(0, \{z\})]. \tag{22}$$

Comparing (22) with equation (15) of the Sompolinsky paper we find that $\varphi(x, H_x(0, \{z\}))$ is equal to a value of the local magnetisation $(m_x\{z\})$ which remains frozen at the time scale t_x. $H_1(0, \{z\})$ is a time-persistent effective field and z is a time-persistent noise, $\langle z(x)z(x')\rangle = \delta(x - x')\dot{q}(x)$.

For $\Psi_x(y, h)$ we have

$$\Psi_x(y, h) = \int_{-\infty}^{\infty} \prod_{y<p<x} \frac{\mathrm{d}z(p)}{(2\pi \dot{q}(p))^{1/2}} \exp\left(-\tfrac{1}{2}\int_y^x \mathrm{d}p \frac{z^2(p)}{\dot{q}(p)}\right) \varphi^2(x, H_x(y, \{z\})).$$

The use of equations (9) and (22) yields $q(x) = \langle m_x^2 \rangle_{\{z\}}$. Relation (19) may be written in the form

$$T^2 = 1 - 2q(1) + [\langle m_1^4 \rangle_{\{z\}}].$$

This relation coincides with the condition of marginal dynamical stability which has been derived by Sompolinsky (1981).

Therefore all the main results of the Sompolinsky dynamic approach may be obtained from the Parisi solution given by equations (7)–(11). There is one distinction: in the framework of the Sompolinsky dynamic approach the functions $q(x)$ and $\Delta(x)$ are supposed to be monotonous functions on the interval [0, 1] while for the Parisi solution there are flat regions (x between 0 and x_0 or between x_1 and 1).

5. Conclusion

In this paper we report an evaluation of the function $q(x)$, magnetisation (m) and free energy (F) by solving the system of self-consistent equations (7)–(11). We prove that there are massless modes. It provides further evidence that the Parisi solution may be an exact solution of the Sherrington–Kirkpatrick model of a spin glass. We showed an equivalence of the solution determined by equations (7)–(11) to the results obtained by Sompolinsky (1981) in the framework of the dynamic approach.

We hope that equations (7)–(11) can be successfully used to investigate magnetic and thermodynamic properties of the Ising spin glass at all temperatures $T < T_c$ and $H \geq 0$.

Appendix

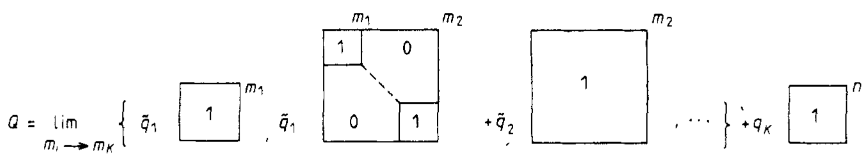
To derive equations (8)–(11) we use a simple algebraic method proposed by Duplantier (1981). In the framework of this method the next representation of the value G (see equation (5))

$$G = \left(\exp \tfrac{1}{2}\beta^2 \sum_{\alpha,\beta} Q_{\alpha\beta} \frac{\partial}{\partial h_\alpha} \frac{\partial}{\partial h_\beta}\right)\left(\prod_\alpha 2 \cosh h_\alpha\right)$$

and the trivial identity

$$\sum_\alpha \frac{\partial}{\partial h_\alpha} f(h_1, \ldots, h_n)|_{h_\alpha = h} = \frac{\mathrm{d}}{\mathrm{d}h} f(h, \ldots, h)$$

are used. Moreover we supposed $Q_{\alpha\alpha} = q_0$. The generic matrix $\hat{Q}$ can be considered as the limit of the series



(A1)

where $\tilde{q}_i = q_{i-1} - q_i$. Now $g(m_i, h)$ and $\tilde{\varphi}(m_i, h)$ are defined as the restricted G and $\partial G/\partial h_\alpha(h_\alpha = h)$ which are calculated for the ith term of the series (A1). We have immediately the recursion:

$$g(m_0, h) = 2\cosh h, \qquad \tilde{\varphi}(m_0, h) = 2\sinh h,$$
$$g(m_1, h) = [\exp \tfrac{1}{2}\beta^2 \tilde{q}_1 (\mathrm{d}^2/\mathrm{d}h^2)][g(m_0, h)]^{m_1/m_0},$$
$$\tilde{\Phi}(m_1, h) = [\exp \tfrac{1}{2}\beta^2 \tilde{q}_1 (\mathrm{d}^2/\mathrm{d}h^2)]\, \{\tilde{\varphi}(m_0, h)[g(m_0, h)]^{m_1/m_0 - 1}\},$$
$$g(m_2, h) = [\exp \tfrac{1}{2}\beta^2 \tilde{q}_2 (\mathrm{d}^2/\mathrm{d}h^2)][g(m_1, h)]^{m_2/m_1},$$
$$\tilde{\varphi}(m_2, h) = [\exp \tfrac{1}{2}\beta^2 \tilde{q}_2 (\mathrm{d}^2/\mathrm{d}h^2)]\{\tilde{\varphi}(m_1, h)[g(m_1, h)]^{m_2/m_1 - 1}\}, \ldots$$
$$G = [\exp \tfrac{1}{2}\beta^2 q_K (\mathrm{d}^2/\mathrm{d}h^2)][g(m_K, h)]^{n/m_K},$$
$$m = [\exp \tfrac{1}{2}\beta^2 q_K (\mathrm{d}^2/\mathrm{d}h^2)]\{\tilde{\varphi}(m_K, h)[g(m_K, h)]^{n/m_K - 1}\}.$$

In the continuous limit $n \to 0$, $m_i = x \in [0, 1]$, $m_{i+1}/m_i = (x - \mathrm{d}x)/x$ and the recursion relation becomes

13031
13032

$$\begin{aligned} g(x, h) &= [\exp \tfrac{1}{2}\beta^2\, \mathrm{d}q(x)(\mathrm{d}^2/\mathrm{d}h^2)][g(x + \mathrm{d}x, h)]^{1 - \mathrm{d}x/x}, \\ \tilde{\varphi}(x, h) &= [\exp \tfrac{1}{2}\beta^2\, \mathrm{d}q(x)(\mathrm{d}^2/\mathrm{d}h^2)]\{\tilde{\varphi}(x + \mathrm{d}x, h)[g(x + \mathrm{d}x, h)]^{-\mathrm{d}x/x}\} \end{aligned} \tag{A2}$$

with $g(1, h) = 2\cosh h$, $\tilde{\varphi}(1, h) = 2\sinh h$. Equations (A2) are equivalent to

$$\partial g/\partial x = -\tfrac{1}{2}\beta^2(\mathrm{d}q/\mathrm{d}x)\partial^2 g/\partial h^2 + (1/x) g \ln g, \tag{A3}$$

$$\partial\tilde{\varphi}/\partial x = -\tfrac{1}{2}\beta^2(\mathrm{d}q/\mathrm{d}x)\partial^2 \tilde{\varphi}/\partial h^2 + \tfrac{1}{2}x\tilde{\varphi} \ln g. \tag{A4}$$

Substituting $g = \exp(xf)$ and $\tilde{\varphi} = \varphi \exp(xf)$ we obtain equations (10) and (11).

Let us consider the calculation of the quantity $\partial^2 G/\partial h_\alpha \partial h_\beta (q_{\alpha\beta} = q_i)$. It is easy to show that the ith term of the series (A1) has the form

$$\tilde{\psi}_{m_i}(m_{i+1}, h) = \tilde{\varphi}^2(m_i, h)[g(m_i, h)]^{m_{i+1}/m_i - 2}. \tag{A5}$$

(A5) immediately gives the recursion:

$$\tilde{\psi}_{m_i}(m_{i+2}, h) = [\exp \tfrac{1}{2}\beta^2 q_{i+2}(\mathrm{d}^2/\mathrm{d}h^2)]\{\tilde{\psi}_{m_i}(m_{i+1}, h)[g(m_{i+1}, h)]^{m_{i+2}/m_{i+1} - 1}\}, \ldots, \tag{A6}$$
$$q_i = \tilde{\psi}_{m_i}(n, h) = [\exp \tfrac{1}{2}\beta^2 q_K (\mathrm{d}^2/\mathrm{d}h^2)]\{\tilde{\psi}_{m_i}(m_K, h)[g(m_K, h)]^{n/m_K - 1}\}.$$

In the limit $n \to 0$ the recursion relation (A6) becomes

$$\tilde{\psi}_x(y, h) = [\exp \tfrac{1}{2}\beta^2 \mathrm{d}q(y)(\mathrm{d}^2/\mathrm{d}h^2)]\{\tilde{\psi}_x(y + \mathrm{d}y, h)[g(y + \mathrm{d}y, h)]^{-\mathrm{d}y/y}\} \tag{A7}$$

with $\tilde{\psi}_x(x, h) = \tilde{\varphi}^2(x, h)/g(x, h)$. Equation (A7) is equivalent to (A4). Substituting $\tilde{\psi}_x(y, h) = \psi(y, h) \exp[yf(y, h)]$, for the function ψ_x we obtain equation (11) with the boundary condition $\psi_x(x, h) = \varphi^2(x, h)$.

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